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BOOK REVIEWS

The Navier–Stokes Equations: An Elementary Functional Analytic Approach.
By H. Sohr. Birkhauser, 2001. 367 pp. ISBN 3 7643 6545 5. sFr 158 or € 104.
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All readers of *JFM* will be very familiar with the Navier–Stokes equations that describe the motion of a viscous fluid. However, some readers may be less familiar with the tools of functional analysis that have become important for approaching some of the significant issues in mathematical fluid dynamics. This book aims to provide a compilation of the major functional analytic results that have been used for more than half a decade in mathematical treatments of the Navier–Stokes equations. Most of the material in the book is available elsewhere in the literature. However, the book serves a useful purpose in the sense that it puts in one volume the functional analysis specifically used in obtaining well-known results concerning the Navier–Stokes equations and gives details of how these results are proved. This being said, we remark that for a book concerned with the motion of fluids it is exceptionally 'dry'. There is no motivation or connection with the physics described by the equations or any implications of the significance of the results obtained using functional analysis. A *JFM* reader who wishes to learn about or refresh knowledge concerning, for example, the use of the Sobolev embedding theorem in the study of the Navier–Stokes equations could find the book a useful technical reference. But such a reader would need to come to the book already convinced of the importance of the Sobolev embedding theorem for mathematical fluid dynamics.

The Navier–Stokes equations studied in this book are the equations for a viscous, constant-density fluid in a general domain $\Omega \subseteq R^n$ with $n=2$ or 3, namely

$$u_t + (u \cdot \nabla)u = -\nabla P + \nu \Delta u + f, \tag{1}$$
$$\nabla \cdot u = 0. \tag{2}$$

The system is considered with initial conditions

$$u(x, 0) = u_0(x) \tag{3}$$

and boundary conditions

$$u|_{\partial\Omega} = 0. \tag{4}$$

In the case that Ω is unbounded (4) is replaced by

$$u(x, t) \to 0 \text{ as } |x| \to \infty. \tag{5}$$

A fundamental problem in analysis is to decide whether smooth, physically reasonable solutions exist for this system for all $t \geqslant 0$, and are unique. We note that in three dimensions this problem is sufficiently challenging that it was chosen as one of the so-called 'million dollar' prize problems offered by the Clay Mathematical Institute (see Fefferman 2000). Formidable mathematical difficulties arise from the subtle complexity of the nonlinearity of equation (1), particularly in three dimensions. Much progress has been made in proving existence, uniqueness and regularity of solutions for the Navier–Stokes system in two dimensions. So far only small, but very interesting, steps have been achieved in the more physically relevant case of three dimensions.

It is reasonable to claim that the mathematical theory of the Navier–Stokes equations, from a 'modern' point of view, started with the seminal work of J. Leray in the 1930s. Motivated by the problem given by (1)–(4), Leray (1933) developed new mathematics, namely the concept of 'weak' solutions to a partial differential equation (PDE) and the concept of topological degree (Leray and Schauder). The notion of weak solutions permits objects in much larger classes than the space of classical functions to be used to describe the motion of a fluid. Such spaces of functions were systematically introduced a little later in 1936 by Sobolev. It is easier to prove the existence of a solution to a PDE, perhaps regular or perhaps singular, in a larger class but such a solution (e.g. a distribution) may not be unique. Leray's theory gives the existence of weak, possibly irregular and possibly non-unique solutions to the Navier–Stokes equations.

This book by Sohr presents the functional analysis required to understand the concept of weak solutions and the related results for the Navier–Stokes equations of J. Leray which were followed by important contributions from a number of mathematicians including O. Ladyzhenskaya, E. Hopf, J. Serrin, J.-L. Lions and T. Kato. The book includes an introduction to the theory of distributions, Sobolev Spaces, certain standard inequalities and compact embeddings. Weak solutions are introduced in the context of the fluid equations first for the steady Navier–Stokes equations, then for the linearized equations and finally for the full system (1)–(4). In the final section of the book what is known as the generalized energy inequality is used to construct and investigate a 'suitable weak solution' for the Navier–Stokes system. If $n=2$, each weak solution is a uniquely determined strong (i.e. classical) solution. If $n=3$, the solution is proved to be strong only under stringent restrictions on the smallness of the data and the smoothness of the boundary.

As Sohr remarks, the literature concerning mathematical fluid dynamics is very extensive and for the sake of a focused exposition he has not attempted to be comprehensive. However his choice of references is perhaps a little too restrictive and many significant contributions to the Navier–Stokes system are omitted. For a broader picture an interested reader might wish to turn to several 'classical' texts such as Ladyzhenskaya (1969) and Temam (republished with additions in 2001) or the recent books of P.-L. Lions (1996) or A. Bertozzi and A. Majda (2001).

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SHORT NOTICES

Flow past Highly Compliant Boundaries and in Collapsible Tubes. Edited by P. W. Carpenter & T. J. Pedley. Kluwer, 2003. 323 pp. ISBN 1-4020-1161-X. £79.00.

This book grew out of an IUTAM Symposium on Flow in Compressible Tubes and Past Compliant Bodies which was held in March 2001 at the University of Warwick. However, it is not a conference proceedings in the normal sense of the phrase. Rather, it consists of a limited number of extended articles by selected participants which aims to sum up the state of the art in this area.

The articles are grouped into three areas: flow in collapsible tubes; instability of flow past compliant walls; and drag and turbulence near compliant walls. There are thirteen papers in total ranging in length from 15 to 35 pages. Attractively produced by Kluwer, this provides a useful insight into the research of flow over compliant boundaries.

Flow Around Circular Cylinders, Vol. 2: Applications. By M. M. Zdravkovich. Oxford University Press, 2003. 613 pp. ISBN 0-19-856561-5. £95.00.

This is the second of two comprehensive books on flow around circular cylinders by M. M. Zdravkovich. Volume 1, which appeared in 1997, was concerned with fundamental aspects of this class of flows. Volume 2, on the other hand, is motivated largely by practical applications. It covers such aspects as: the influence of aspect ratio; surface roughness; wall proximity; boundary layer control; yawed cylinders; and clusters of cylinders. At almost 600 pp it provides detailed discussion of flow around cylinders and contains a wealth of references. It represents a useful reference source in a single volume.